

MATH 281 Workbook 2 – Project 2

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October 20, 2025

Statement

This report summarizes our work and analysis for Project 2.

Problem 1 – Annie Wang

Problem 2 – Annie Wang

Problem 3 – Ethan Raupers

Problem 4 – Iliana Balaskas

1 Question 1: Spectral Derivatives and Error vs. Δx

Wavenumber–space coefficients

Let $f(x)$ be L -periodic with Fourier coefficients c_n at wavenumbers $k_n = \frac{2\pi n}{L}$ (in code: `kx = 2*np.pi*fftfreq(N, d=dx)`). If $\tilde{c}_n^{(m)}$ denotes the Fourier coefficients of the m -th derivative, then

$$\boxed{\tilde{c}_n^{(m)} = (ik_n)^m c_n}.$$

Explicitly,

$$\begin{aligned} \text{1st: } \tilde{c}_n^{(1)} &= (ik_n)c_n, \\ \text{2nd: } \tilde{c}_n^{(2)} &= -k_n^2 c_n, \\ \text{3rd: } \tilde{c}_n^{(3)} &= -i k_n^3 c_n, \\ \text{4th: } \tilde{c}_n^{(4)} &= k_n^4 c_n. \end{aligned}$$

The physical-space derivatives are obtained by inverse FFT:

$$f^{(m)}(x) = \text{IFFT}((ik)^m \text{FFT}(f)) \big|_{\text{real}}.$$

Methodology

- Test function: $y = \sin x$ on $[0, 2\pi]$ with $N = 256$ (periodic grid). Analytical derivatives: $y' = \cos x$, $y'' = -\sin x$, $y^{(3)} = -\cos x$, $y^{(4)} = \sin x$.
- Spectral derivatives computed as $\text{IFFT}((ik)^m \text{FFT}(y))$ for $m = 1, \dots, 4$.
- Convergence test for first derivative: $y = \sin(mx)$ with $m = 1$ and $m = 10$ (“Nmodes = 1, 10”). Compared: spectral, centered FD2, centered FD4. Error metric: $L^2 = \sqrt{\langle (y_x^{\text{num}} - y_x^{\text{true}})^2 \rangle}$.

Results

(a) Accuracy for fixed grid and multiple derivative orders. Figure 1 compares the true and spectrally computed 1st–4th derivatives of $\sin x$ on a fine grid ($N = 256$). The curves are visually indistinguishable: any small difference lies at rounding level. This is expected because, for a single resolvable Fourier mode, multiplying each Fourier coefficient by $(ik)^m$ produces the exact derivative (up to machine precision).

(b) Error vs. Δx . Figures 2–3 show L^2 error of the first derivative versus grid spacing for Nmodes = 1 and 10, comparing FD2, FD4, and spectral differentiation.

Observed behavior.

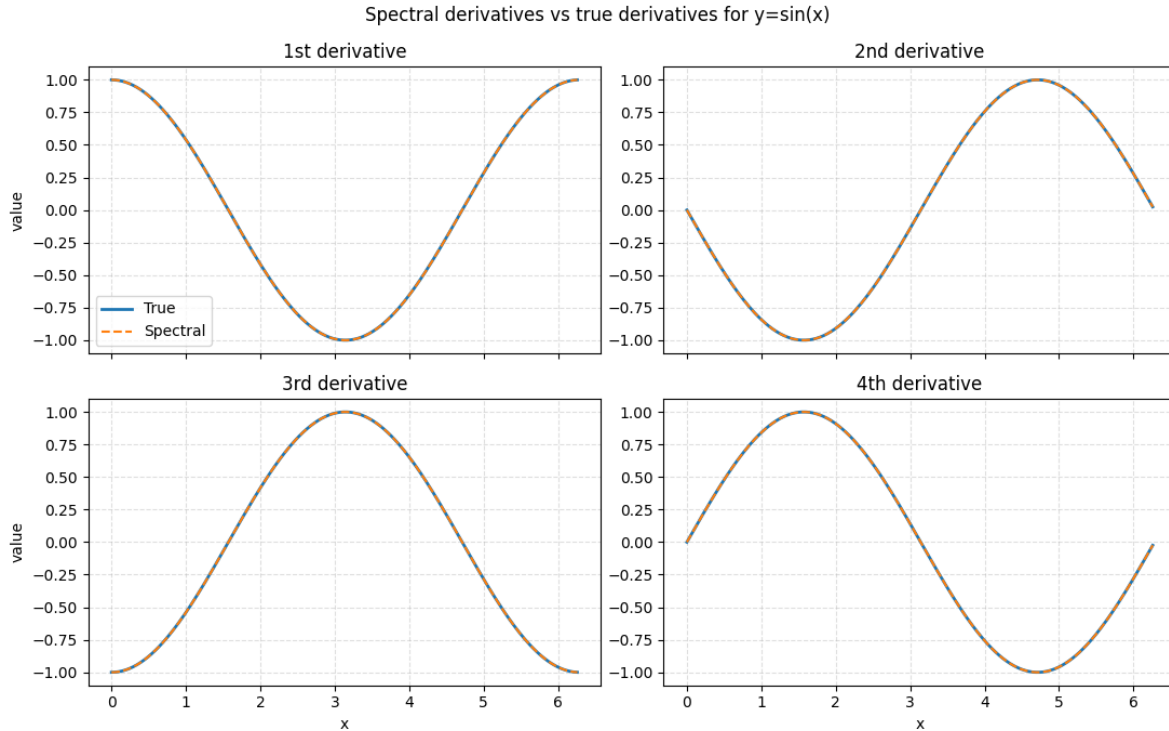


Figure 1: Spectral derivatives (dashed) versus analytical truth (solid) for $y = \sin x$. All four derivative orders match to numerical precision.

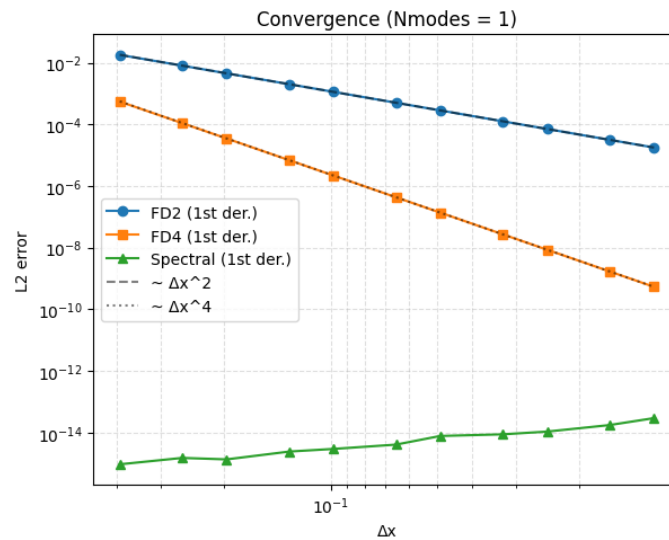
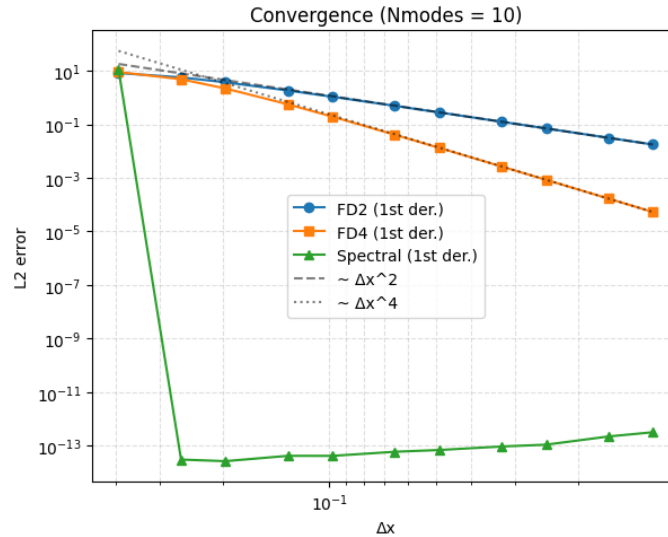


Figure 2: Convergence of first-derivative error ($N_{\text{modes}} = 1$).

- FD2 errors scale as $\mathcal{O}(\Delta x^2)$, and FD4 errors as $\mathcal{O}(\Delta x^4)$, matching theoretical slopes.
- Spectral errors remain essentially flat at the machine-epsilon level ($\sim 10^{-14}$) once the mode is resolved (wavenumber $\ll$ Nyquist).



3. **Resolution dependence.** For higher mode numbers (shorter wavelengths), both FD and spectral methods require adequate resolution, but once resolved, the spectral method is orders of magnitude more accurate.
4. **Practical takeaway.** Use spectral differentiation when (i) the problem is periodic, and (ii) the grid resolves the highest wavenumber in the signal. Otherwise, finite-difference methods deliver predictable polynomial convergence.

Implementation notes. Using cell-centered grids for the convergence sweep helps avoid endpoint phase shifts. Applying `.real` after the inverse FFT discards tiny imaginary round-off components. To prevent aliasing, ensure $k_{\max} \geq N_{\text{modes}}$.

2 Question 2: Measuring Spectral Power

Fourier representation and definition of spectral power

For a real-valued, periodic function $f(x)$ on a domain of length L , its Fourier expansion reads

$$f(x) = \sum_{n=-N/2}^{N/2-1} c_n e^{ik_n x}, \quad k_n = \frac{2\pi n}{L}.$$

The ****power spectrum**** is defined as the squared magnitude of the Fourier coefficients:

$$P(k_n) = |c_n|^2,$$

which represents the distribution of variance (or energy) across wavenumbers. Large k correspond to small physical length scales, while small k correspond to large scales.

Since $f(x)$ is real, $c_{-n} = c_n^*$, so it is sufficient to plot only the ****one-sided spectrum**** ($k \geq 0$):

$$P^+(k_n) = \begin{cases} |c_0|^2, & n = 0, \\ 2|c_n|^2, & n > 0. \end{cases}$$

Functions analyzed

We compute and compare the spectral power for three representative signals on $x \in [-16, 16]$ ($L = 32$):

$$f_1(x) = e^{-x^2}, \quad (\text{smooth Gaussian}),$$

$$f_2(x) = \tanh(5 - |x|), \quad (\text{steep transition}),$$

$$f_3(x) = \cos[\pi(x/2)^2], \quad (\text{chirp: frequency increasing with } |x|).$$

All functions are sampled with $N = 4096$ points to ensure negligible aliasing.

Numerical method

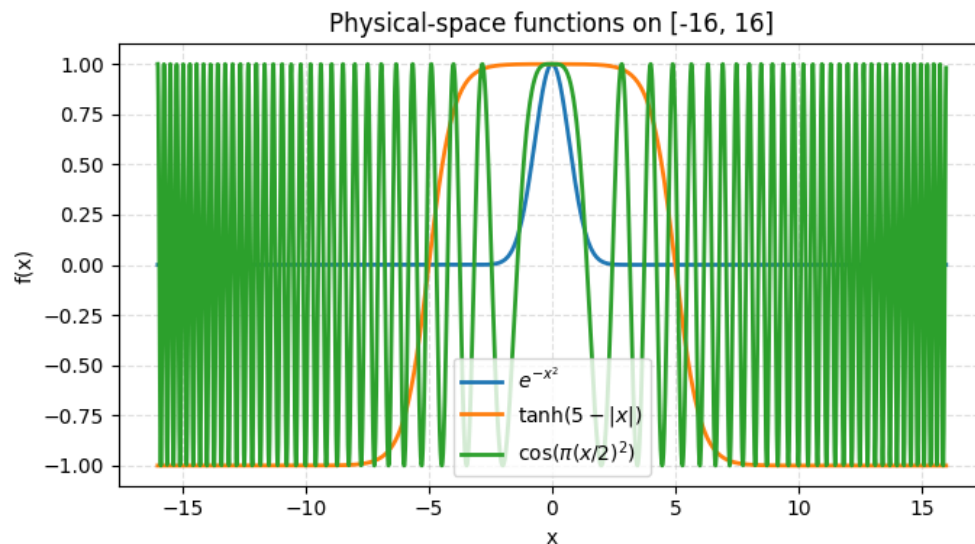
The discrete Fourier transform (DFT) and the corresponding wavenumber array were obtained using `numpy.fft`:

$$c_n = \frac{1}{N} \text{fft}(f), \quad k_n = \frac{2\pi}{L} \text{fftfreq}(N, d = \Delta x).$$

The one-sided power spectrum $P^+(k_n) = 2|c_n|^2$ (for $n > 0$) was then plotted on logarithmic axes to highlight differences in spectral decay.

Results and interpretation

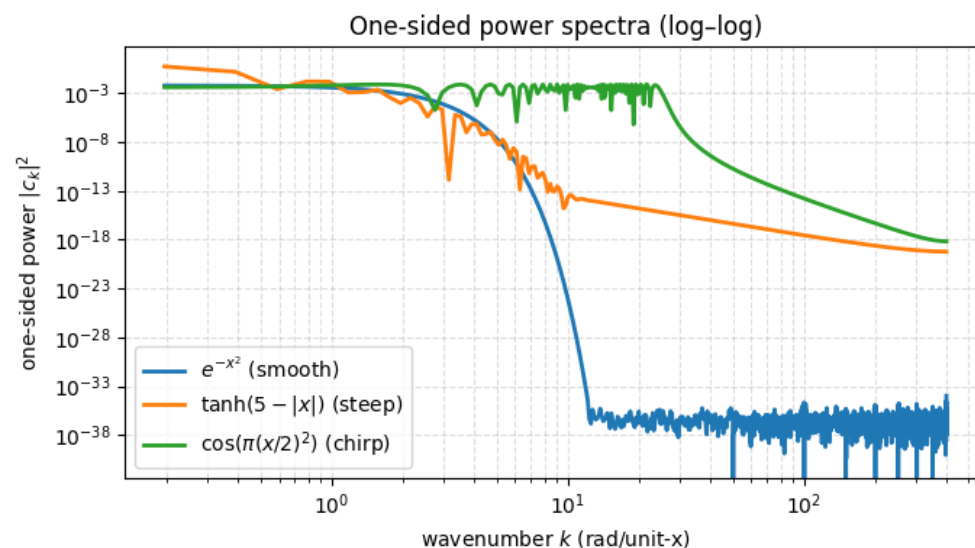
(a) Physical-space functions.



The three test functions represent distinct degrees of smoothness:

- e^{-x^2} : infinitely differentiable and strongly localized;
- $\tanh(5 - |x|)$: continuous but with sharp slope changes at $x \approx \pm 5$;
- $\cos[\pi(x/2)^2]$: non-stationary "chirp," with wavelength decreasing as $|x|$ grows.

(b) One-sided power spectra (log-log).



The resulting spectra $P^+(k)$ clearly reflect each function's spatial character:

Gaussian e^{-x^2} : The blue curve decays *faster than any power law*—essentially exponentially—indicating that almost all energy is concentrated in the lowest wavenumbers. This is the hallmark of an analytic, infinitely smooth signal.

tanh $(5 - |x|)$: The orange curve decays approximately as a power law $P \sim k^{-p}$ with $p \approx 2-4$, retaining substantial power at high k . The slow decay corresponds to the sharp transitions at $x \approx \pm 5$: a localized steep edge produces broadband frequency content.

Chirp $\cos[\pi(x/2)^2]$: The green curve spans a wide k -range without a single slope. Because the instantaneous frequency increases with $|x|$, the function contains both large- and small-scale features, resulting in a broad, structured spectrum.

Conclusions

- Smoothness in x -space translates to rapid decay in k -space. $\Rightarrow$ The Gaussian, being analytic, exhibits an extremely narrow spectrum.
- Non-smooth or sharply varying signals carry significant high-wavenumber power. $\Rightarrow$ The tanh function has a broad power tail.
- Non-stationary signals spread their power over a wide range of k . $\Rightarrow$ The chirp contains contributions from multiple characteristic scales.

These results together illustrate how the power spectrum quantitatively captures a function's scale content: broad spectra correspond to fine spatial structures, while rapidly decaying spectra signify smooth, large-scale variations.

3 Question 3: Heat Equation Analysis

3.1 Part A

Below are the solutions for `heat.npz`. This file gives numerical solutions to the equation

$$\partial_t u(x, t) = \partial_{xx} u(x, t) \quad (1)$$

with initial condition

$$u(x, t = 0) = \exp\left[-\left(\frac{x}{0.5}\right)^2\right] + 0.25 \sin(5\pi x), \quad (2)$$

for

$$x \in [-5, 5], \quad t \in [0, 1].$$

Figure 4 depicts the power spectrum for times “t”. This power spectrum graphs the power $|c_n|^2$ over the positive wavenumber values. Figure 5 depicts the physical field solution for the spectral graphs. The physical plot shows the solution $u(x, t)$ in space for the same time.

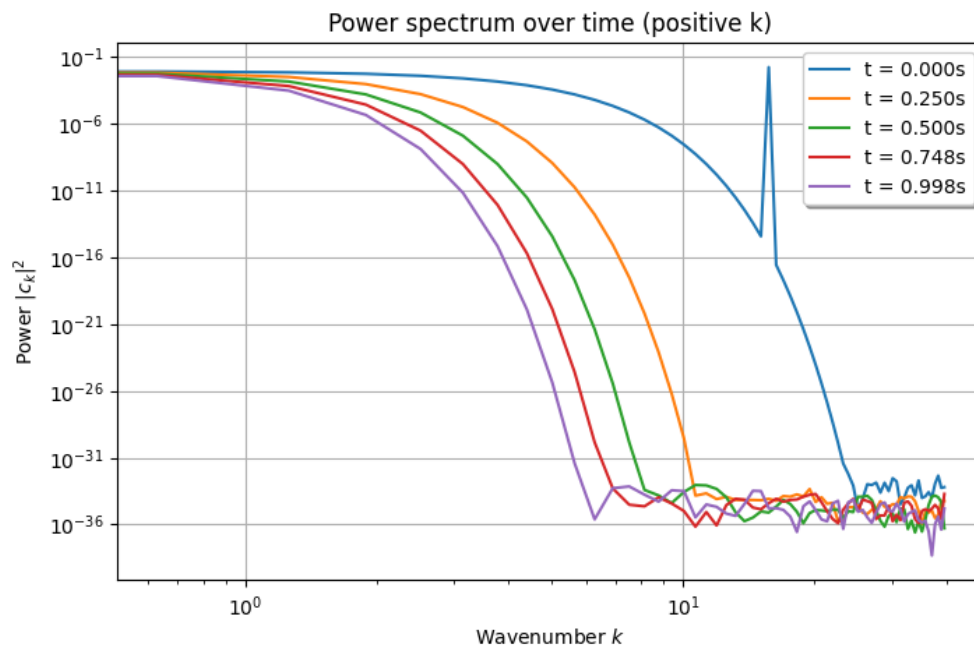


Figure 4: Power spectrum over time (positive k) for `heat.npz`.

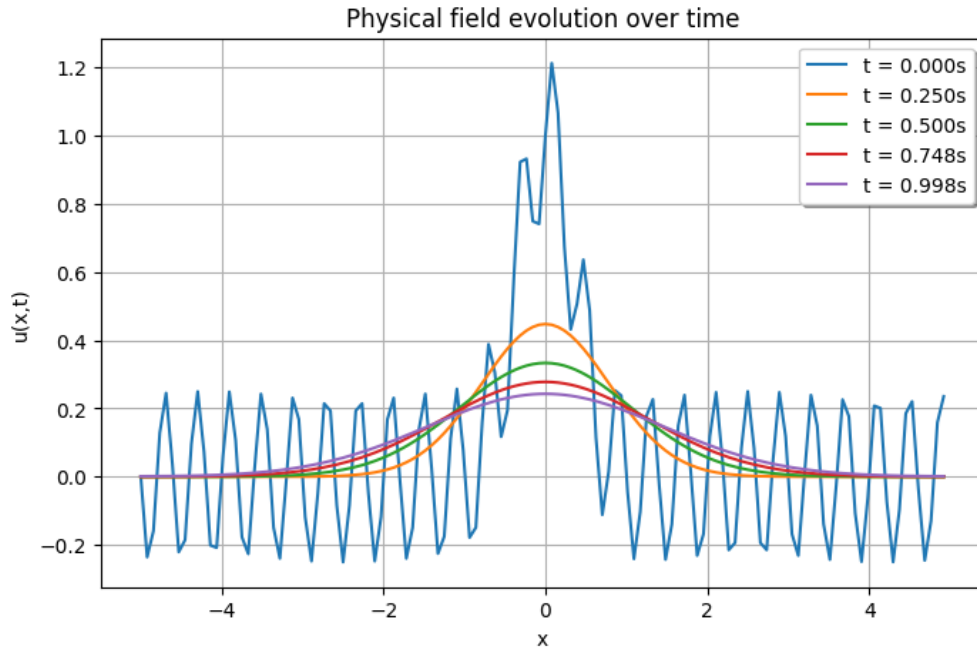


Figure 5: Physical field evolution over time for `heat.npz`.

Within the power spectrum, we can see that lower wavenumbers correspond to higher amounts of power. This provides a smooth-like structure at all time values. As the wavenumber increases for all t , we can see a decrease in power. The power spectrum demonstrates energy loss in each mode with an increasing wavenumber. As time increases, we can see a sharper decrease in energy within the wavenumbers. This manifests itself in the physical plot by smoothing its features.

In $t = 0$ s for the physical plot, we can see sinusoidal features except for $x = 0$. Here we can see a gaussian bell curve structure. As time evolves, the higher wavenumber manifests itself as a smoothing of the features of the physical plot. This is in corroboration with the power spectrum that loses energy at higher times and wavenumbers.

We can consider the initial conditions to describe these effects. In the initial condition, we have both an exponential decay term and a sinusoidal term. This would describe the shape described in $t = 0$ s. The graph is dominated by sinusoidal structure except at the origin. This is where $\sin(0) = 0$ and only the gaussian component would remain. Examining the energy loss, we can explain the phenomenon with the exponential decay rate of the initial condition.

Note: The small irregularities and oscillations visible at high wavenumbers within Figure 4 are a result of numerical noise in the code's computations. These artifacts do not represent physical behavior of the solution and are a result of the discrete Fourier transform (FFT) and numerical rounding errors in Python.

3.1 Part B

To find the spectral power for the given ODE, we can substitute our PDE into our ODE and solve. This gives us

$$\begin{aligned}
 u(x, t) &= \sum_n \hat{u}_n(t) e^{ik_n x} \\
 \frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} \\
 \frac{\partial}{\partial t} \left(\sum_n \hat{u}_n e^{ik_n x} \right) &= \sum_n \hat{u}_n \frac{\partial^2}{\partial x^2} (e^{ik_n x}) \\
 \sum_n \frac{d\hat{u}_n}{dt} e^{ik_n x} &= \sum_n \hat{u}_n (-k_n^2) e^{ik_n x} \\
 \frac{d\hat{u}_n}{dt} &= -k_n^2 \hat{u}_n \\
 \frac{d\hat{u}_n}{\hat{u}_n} &= -k_n^2 dt \\
 \Rightarrow \hat{u}_n &= \hat{u}_n(0) e^{-k_n^2 t}
 \end{aligned}$$

The spectral power is defined as

$$\begin{aligned}
 P_n(t) &= |\hat{u}_n(t)|^2 \\
 &= |\hat{u}_n(0)|^2 e^{-2k_n^2 t}
 \end{aligned}$$

and its time derivative is

$$\frac{dP_n}{dt} = -2k_n^2 P_n.$$

We can use the specified spectral power equation to graph the time evolution of the spectral power as the theorized value versus the simulated value as described by $|c_n|^2$.

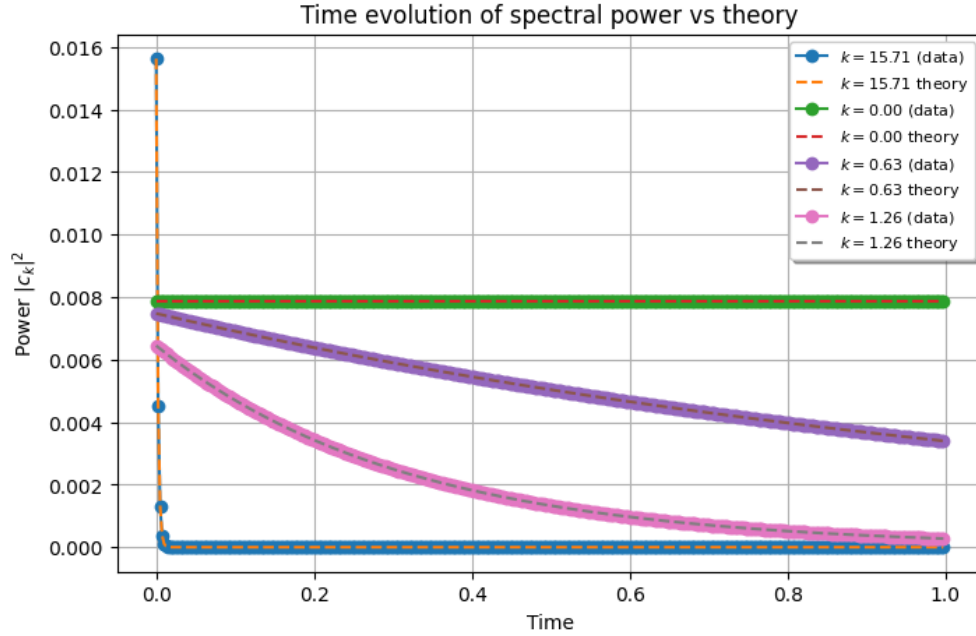


Figure 6: Theorized time evolution vs data value.

We can see that when plotted, our simulated values mirror our theorized values for the spectral power of the ODE. The numerical results show that higher wavenumbers decay faster (steeper slope on the log plot), exactly matching the theoretical exponential behavior. This again confirms that larger wavenumber lose energy at a considerable rate compared to lower wavenumbers.

3.2 Part C

Considering Figure 4, we understand that the slopes of the graphs would tell us how the power spectrum would change as a function of the wavenumber. If we see a negative slope, there would be a decrease in energy in a larger wavenumber producing smooth structures within the physical field as time evolves. Positive slopes would tell us that there is more energy at larger wavenumber, and we would see more sinusoidal structure within the physical field as time evolves. In our case, we can see a negative slope, meaning we would see a smoothing of the physical field structure in Figure 5. This is seen as time evolves. The features smooth from the jagged Gaussian bump to a smooth curve. This stays consistent with our negative slope as we lose energy as time evolves, causing the curve to flatten.

4 Question 4: Viscous Burgers Equation Analysis

4.1 Part A

Below are the solutions for `heat.npz`. This file gives numerical solutions to the equation

$$\partial_t u(x, t) = -u \partial_x u + \nu \partial_{xx} u \quad (3)$$

with initial condition

$$u(x, t = 0) = 0.1 \exp \left[- \left(\frac{x}{0.75} \right)^2 \right] \quad (4)$$

for

$$x \in [-10, 10], \quad t \geq 0.$$

Below are two images depicting the plots for Part A — the physical field and the power spectrum for the numerical solution of Burgers' equation with an added viscosity term.

Below are two images depicting the two plots for Part A, the physical field and the power spectrum for the numerical solution of Burgers equation with an added viscosity term.

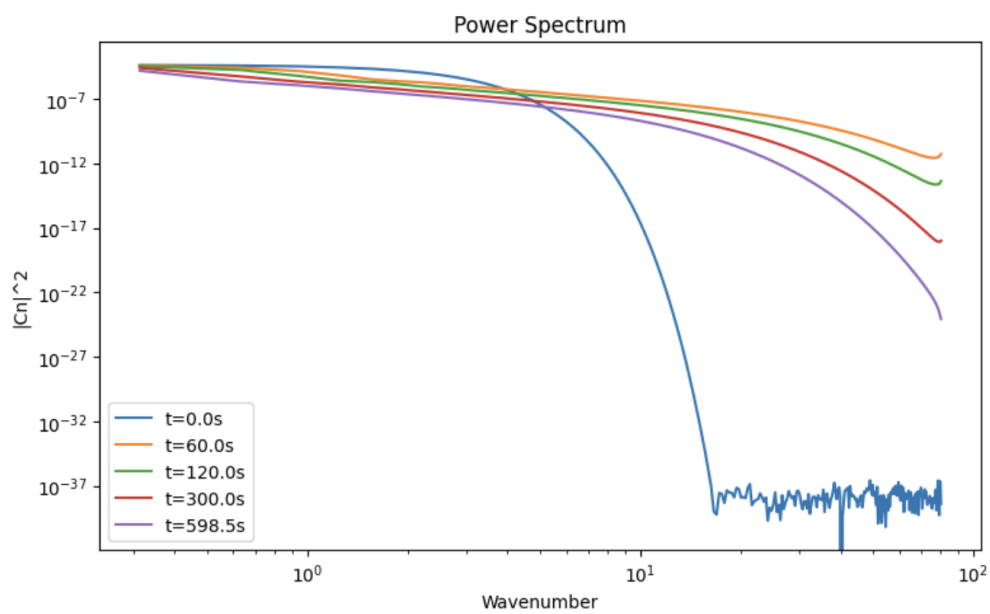


Figure 7: Power spectrum plot for Burgers' equation.

The power spectrum plot shows how $|C_N|^2$ behaves at different wavenumbers, k , at various times for Burgers' equation. At low wavenumbers, there is the highest amount of power resulting in a smooth curve for low wavenumbers at all t values. Small wavenumbers mean longer wavelengths; since we have such high energy at such low frequencies, we know

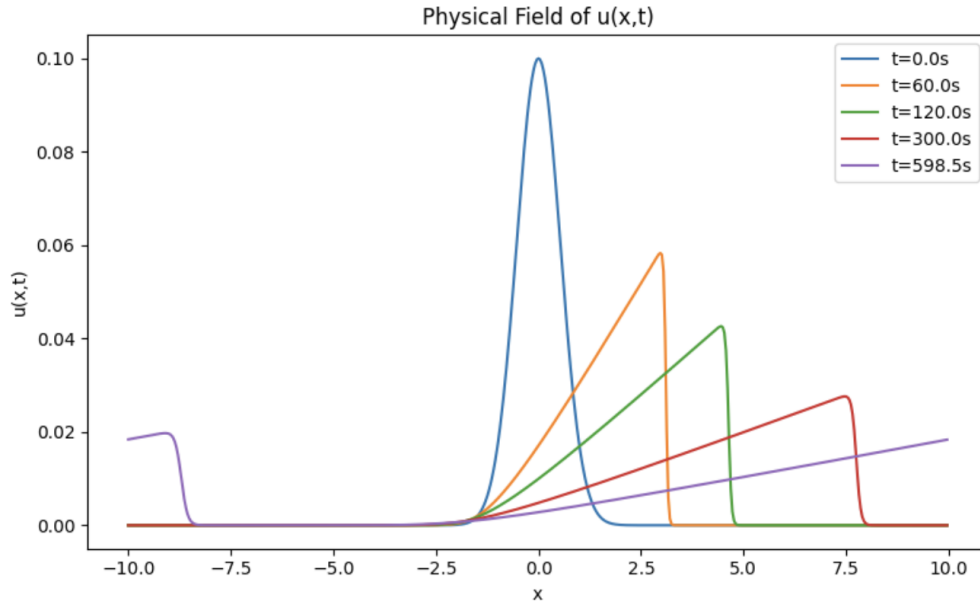


Figure 8: Physical field plot for Burgers' equation.

that the majority of the power results from the start of the graph (where $|C_n|^2$ is the highest), rather than at high wavenumbers. At higher wavenumbers, and early t values (specifically $t = 0$ in the plot), the curve drops off. We know from Burgers' equation that the viscosity term falls off proportional to the wavenumber squared, as expected from the viscosity term. It therefore makes sense that at high wavenumbers there is a low amount of power. The uneven oscillations at high wavenumbers for $t = 0$ are likely due to noise from the plot calculations made.

Looking at the other four time values, it can be seen that with smaller t values, there is more energy at high wavenumbers than at very high wavenumbers; it can be seen for $t = 598.5$ s that the curve starts to drop significantly more at high wavenumbers compared to the curve for $t = 60$ s. This indicates that as time progresses, the power at high wavenumbers is being lost faster due to the viscosity term while the lower wavenumbers, while the energy decays more slowly.

In the physical field plot, we can see that at $t = 0$, there is a Gaussian bell curve centered at $x = 0$. As time increases, the curves shift right, slowly losing their Gaussian curve shape till at $t = 598.5$ s the line looks almost linear. This is a result of the viscosity term causing the function to decrease in amplitude and flatten out across x . The rightward motion of the peak is due to the nonlinear term in Burgers' equation that pushes the wave in the positive x direction. Over time, the two terms in the Burgers' equation result in the sharp edge of the curve being smoothed out and effectively decreased.

Putting all of this together, it can be concluded that as t increases, the energy at high

wavenumbers decreases due to the damping term, resulting in a smoother curve in our physical field. It can be seen that the viscosity term dampens $u(x, t)$ for all the curves, but it can be specifically seen in the power spectrum plot that high wavenumbers dampen a lot faster. At early t values, the nonlinear term briefly transfers energy from low wavenumbers to high wavenumbers, creating the sharpness observed in the physical field plot; as t increases, the viscosity term dominates, resulting in a loss of energy at higher wavenumbers. This all follows the initial condition as stated earlier, because as we expect, at $-10 < x < 10$, the small wavenumber values with high energy result in a smooth Gaussian curve in the physical field. As time increases, the viscosity term decreases the energy at high wavenumbers, resulting in a physical field curve that first sharpens and then smooths across x . Looking specifically at $t = 598.5$ s, we can see that its steepness in the power spectrum plot results in the smoothest physical field.

Note: The small irregularities and oscillations visible at high wavenumbers, specifically for the $t = 0$ curve in the power spectrum plot, are a result of numerical noise in the code's computations. These artifacts do not represent physical behavior of the solution and are a result of the discrete Fourier transform (FFT) and numerical rounding errors in Python.

4.2 Part B

Considering the spectrum in log-log space would tell us how the power spectrum changes as the wavenumber changes. A positive slope implies that the power increases as the wavenumber increases. This would result in u having sharp changes and a much rougher curve. A negative slope would mean that the power spectrum decreases with increasing wavenumbers. That would mean that all the power would be at smaller wavenumbers (just as we saw with $t = 0$ on the graphs from part (a)). This negative slope would create a much smoother curve with a slower change in shape. This is once again similar to what we saw in our power spectrum plot for Part A.

Further, at small t values (not $t = 0$), there will be more power at higher wavenumbers (following the initial condition), compared to later t values where a decrease in the power, and thereby the slope, will occur. We therefore know that as time increases, the slope will become more negative.

This all plays a role in how the physical-space field behaves. Since we will see a more negative slope over time, we should expect to see a smoother curve in the physical field. Low wavenumber features will dominate the behavior of the graph since there is such low energy at higher wavenumbers.

4.3 Part C

Heat Equation from Problem 3:

$$\begin{aligned}\partial_t u(x, t) &= \partial_{xx} u(x, t), \\ u(x, t = 0) &= \exp\left[-\left(\frac{x}{0.5}\right)^2\right] + 0.25 \sin(5\pi x), \\ x &\in [-5, 5], \\ t &\in [0, 1].\end{aligned}$$

Burgers Equation from Problem 4:

$$\begin{aligned}\partial_t u(x, t) &= -u \partial_x u + \nu \partial_{xx} u, \\ u(x, t = 0) &= 0.1 \exp\left[-\left(\frac{x}{0.75}\right)^2\right], \\ x &\in [-10, 10], \\ t &\geq 0.\end{aligned}$$

Comparing the results of Problems 3 and 4, just by observing the equations, we can tell that the qualitative difference in the spectral plotting of the two PDE solutions is due to the nonlinear term present in Burgers equation. Since the heat equation does not have such a nonlinear term, its power spectrum decays at a constant rate with respect to time. The diffusion term carries out all of the dampening applied to the curves, which is what results in the smooth decrease of the curves for all times, t .

Conversely, the nonlinear term in Burger's equation causes different results for how energy moves from lower to higher wavenumbers at various time values. This term, as discussed in Parts (a) and (b), causes a flatter, smoother power spectrum from low to high wave numbers; as time increases, the viscosity term dominates the function, and at high t values, the power for high wavelengths begins to decrease again, just as we observe in Problem 3.

These results are also why, in the physical field graph for the heat equation, at all t values, the curve stays centered about $x=0$. While, in contrast, for Burgers equation, the nonlinear term “shifts” the solution rightward as discussed earlier. The heat equation distributes symmetrically from its starting point without any of the changes from steepness to smoothness that are observed in the physical field graph seen in Problem 4.