

MATH 281 Workbook 3 – Project 3

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Statement

This report summarizes our work and analysis for Project 3.

Problem 1 – Annie Wang

Problem 2 – Annie Wang

Problem 3 – Ethan Raupers

Problem 4 – Illiana Balaskas

Problem 5 – Illiana Balaskas

1 Question 1: PDE Analysis

We analyze the PDE

$$u_t + x u_x = 0, \quad x \in (-L, L), \quad u(x, 0) = u_0(x),$$

using the method of characteristics. No numerical simulation is required.

(a) Total derivative along a trajectory

Consider a trajectory $x = x(t)$ and the composite function $u(x(t), t)$. By the chain rule,

$$\frac{d}{dt}u(x(t), t) = u_t(x(t), t) + \frac{dx}{dt}u_x(x(t), t).$$

Hence, as an operator identity,

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x}.$$

(b) Characteristic curves

The PDE is

$$u_t + x u_x = 0.$$

To write this as a total time derivative, we choose the trajectory such that

$$\frac{dx}{dt} = x.$$

Then

$$\frac{d}{dt}u(x(t), t) = u_t + x u_x = 0,$$

so u is constant along each characteristic.

Solving the characteristic ODE:

$$\frac{dx}{dt} = x \implies x(t) = C e^t.$$

If the characteristic passes through $(x_0, 0)$, then $x(0) = x_0$ gives $C = x_0$, so

$$\boxed{x(t) = x_0 e^t.}$$

Thus, along each characteristic $u(x(t), t) = u_0(x_0)$ remains constant.

(c) Separation of characteristics and system behaviour

Consider characteristics through $(x_0, 0)$ and $(x_1, 0)$, with $x_1 > x_0$. Their trajectories are

$$x_0(t) = x_0 e^t, \quad x_1(t) = x_1 e^t.$$

The distance between the two is

$$\Delta x(t) = x_1(t) - x_0(t) = (x_1 - x_0)e^t.$$

Therefore,

$$\boxed{\Delta x(t) \text{ grows exponentially in } t.}$$

Conclusion: Characteristics diverge—the flow “stretches” the domain outward from the origin. No characteristic crossing occurs, so the solution remains smooth and no shocks form. Information carried by u is transported outward and spread over larger spatial scales.

(d) CFL restriction for a numerical scheme

For an explicit advection scheme, a standard CFL condition is

$$\frac{|a(x)| \Delta t}{\Delta x} \leq C_{\max}, \quad a(x) = x.$$

Since $|x| \leq L$ on the domain $(-L, L)$, a uniform safe restriction is

$$\boxed{\Delta t \leq C_{\max} \frac{\Delta x}{L}.}$$

This ensures that information anywhere in the domain travels no more than one grid cell per step, which is the standard requirement for stability in explicit advection schemes.

(e) Long-time limit of the solution

From part (b), along (x, t) we have

$$x(t) = x_0 e^t \implies x_0 = x e^{-t},$$

so the solution can be written explicitly as

$$u(x, t) = u_0(x e^{-t}).$$

As $t \rightarrow \infty$,

$$xe^{-t} \rightarrow 0,$$

and u_0 is smooth, hence continuous at 0. Therefore,

$$\boxed{u(x, t) \xrightarrow[t \rightarrow \infty]{} u_0(0), \quad \text{uniformly for all } x \in (-L, L).}$$

Interpretation: Since characteristics diverge exponentially, a fixed Eulerian point x corresponds to an exponentially shrinking initial location xe^{-t} near the origin. Thus all interior points eventually “sample” only $u_0(0)$, yielding a spatially uniform long-time state. Outflow boundaries guarantee that no information re-enters the domain.

2 Question 2: Linear Advection

We consider the linear advection equation

$$u_t + U_0 u_x = 0, \quad U_0 = 0.1,$$

on the periodic domain $x \in [-L_x/2, L_x/2]$, with $L_x = 10$, over $t \in [0, 50]$. The initial condition is

$$u(x, 0) = \sin\left(\frac{4\pi x}{L_x}\right) + 0.25 \sin\left(\frac{8\pi x}{L_x}\right).$$

(a) Dispersion relation and Fourier interpretation

We use the plane-wave ansatz

$$u(x, t) = e^{i(kx - \omega t)}.$$

Differentiating gives

$$u_t = -i\omega u, \quad u_x = ik u.$$

Substituting into the PDE,

$$-i\omega u + U_0(iku) = 0 \quad \Rightarrow \quad \omega = U_0 k.$$

Thus the dispersion relation is

$$\boxed{\omega(k) = U_0 k},$$

which is purely real.

Substituting back,

$$u(x, t) = e^{i(kx - U_0 kt)} = e^{ik(x - U_0 t)}.$$

The exponent has no real part, so the amplitude of each Fourier mode is constant. Each mode is simply translated with phase speed

$$c_{\text{phase}} = \frac{\omega}{k} = U_0,$$

independent of k . Hence the system is *non-dispersive*: all Fourier components propagate at the same speed with no growth or decay.

Because a general periodic initial condition can be written as a Fourier series, this analysis implies:

$$u(x, t) = u_0(x - U_0 t),$$

i.e. a rigid translation of the initial profile.

(b) Numerical method, parameters, and simulation results

We use the second-order Lax–Wendroff scheme on a periodic grid:

$$u_j^{n+1} = u_j^n - \frac{C}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{C^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

with CFL number

$$C = \frac{U_0 \Delta t}{\Delta x}.$$

Choice of spatial resolution. The initial condition contains two wavelengths:

$$\lambda_1 = \frac{L_x}{2} = 5, \quad \lambda_2 = \frac{L_x}{4} = 2.5.$$

The shortest wavelength is $\lambda_{\min} = 2.5$. With $N_x = 256$,

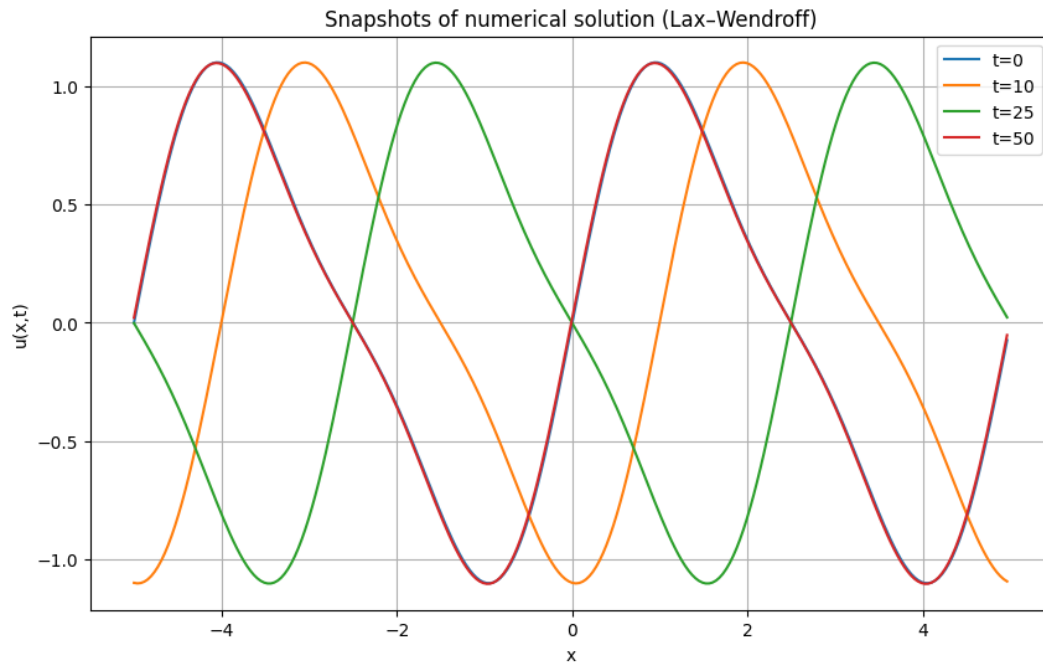
$$\Delta x = \frac{L_x}{N_x} = 0.0390625, \quad \frac{\lambda_{\min}}{\Delta x} \approx 64 \text{ points/wavelength},$$

which cleanly resolves all spatial features.

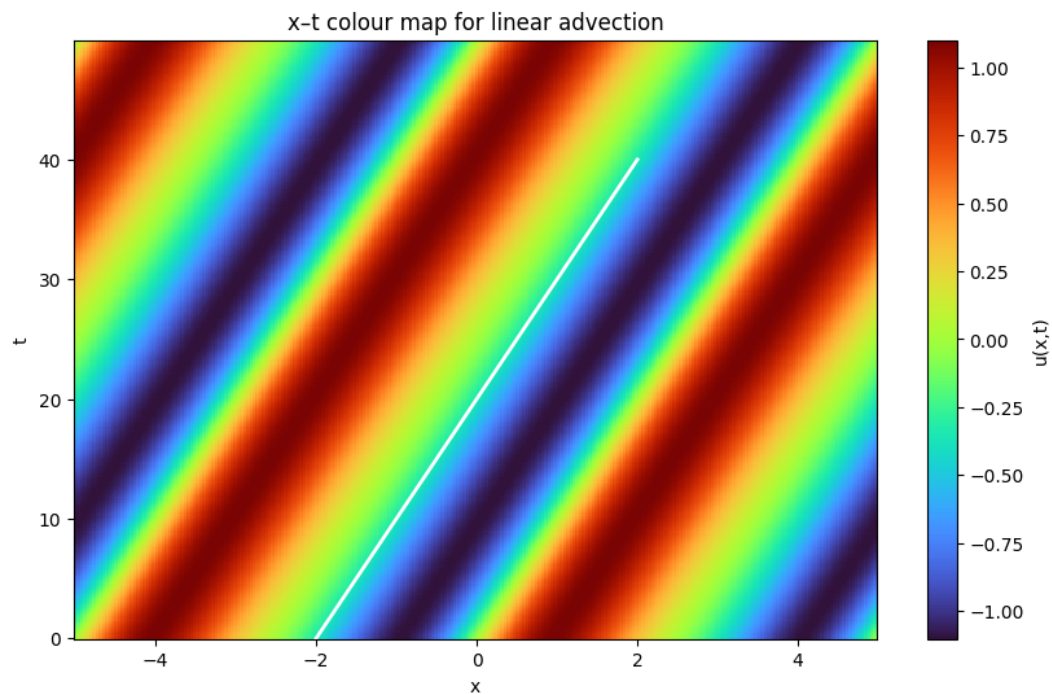
Choice of time step. We pick $C = 0.3$, safely within the stability limit $|C| \leq 1$. Thus

$$\Delta t = C \frac{\Delta x}{U_0} = 0.3 \frac{0.0390625}{0.1} \approx 0.117.$$

Qualitative observations. Snapshots of the numerical solution at several times show that the wave profile is perfectly preserved and translated to the right with speed U_0 , in agreement with part (a).



A space-time colour map further shows straight diagonal stripes, whose slope matches the theoretical advection speed.



These results agree exactly with the analytical prediction that the PDE performs a rigid translation with no distortion.

(c) RMS error in time

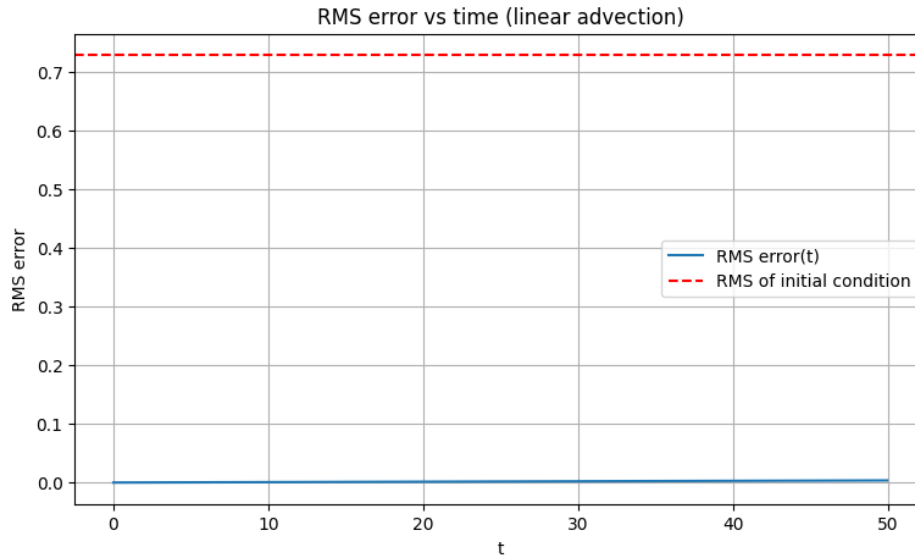
The exact solution is

$$u_{\text{exact}}(x, t) = \sin\left(\frac{4\pi(x - U_0 t)}{L_x}\right) + 0.25 \sin\left(\frac{8\pi(x - U_0 t)}{L_x}\right).$$

The RMS error is computed as

$$\text{RMS}(t) = \sqrt{\frac{1}{N_x} \sum_{j=1}^{N_x} (u_j^{\text{num}}(t) - u_j^{\text{exact}}(t))^2}.$$

The RMS error remains extremely small and almost flat over time, confirming stability and accuracy of the numerical method.



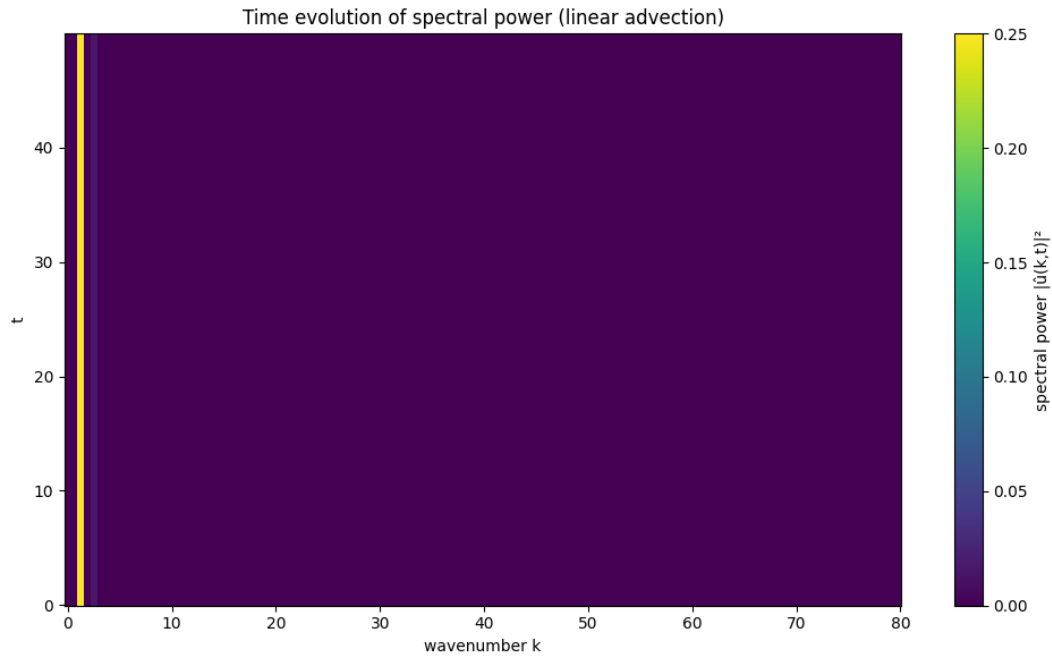
Because Lax–Wendroff is second-order and the solution is a pure translation, this behaviour is expected.

(d) Time evolution of spectral power

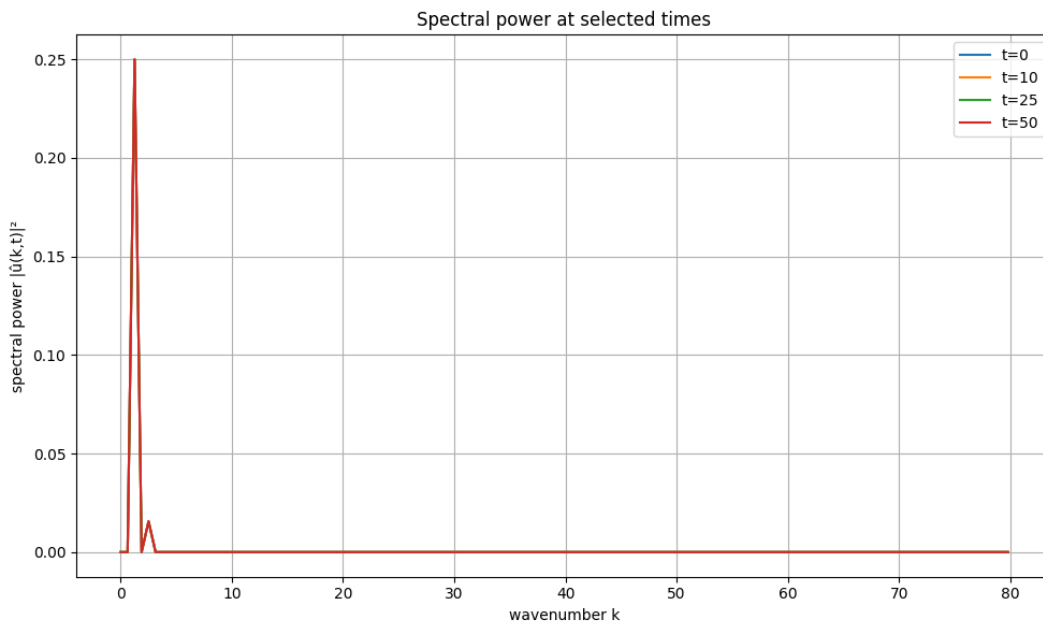
For each stored time, we compute the FFT of $u(x, t)$ and plot the spectral power

$$|\hat{u}(k, t)|^2.$$

The colour map shows two strong stationary bands corresponding to the two initial Fourier modes. Their power remains essentially constant in time.



Line plots at selected times (0, 10, 25, 50) lie almost exactly on top of each other:



This agrees with the analytical result from (a): since $\omega(k) = U_0 k$ is real, the amplitude of each Fourier mode is preserved.

(e) Energy analysis

Define the energy

$$E(t) = \int_{-L_x/2}^{L_x/2} u(x, t)^2 dx.$$

(i) Derivation of ODE for $E(t)$. Differentiate:

$$\frac{dE}{dt} = \int 2u u_t dx = \int 2u(-U_0 u_x) dx = -2U_0 \int uu_x dx.$$

Integration by parts gives

$$\int uu_x dx = \frac{1}{2}u^2 \Big|_{-L_x/2}^{L_x/2} = 0$$

because the solution is periodic. Thus

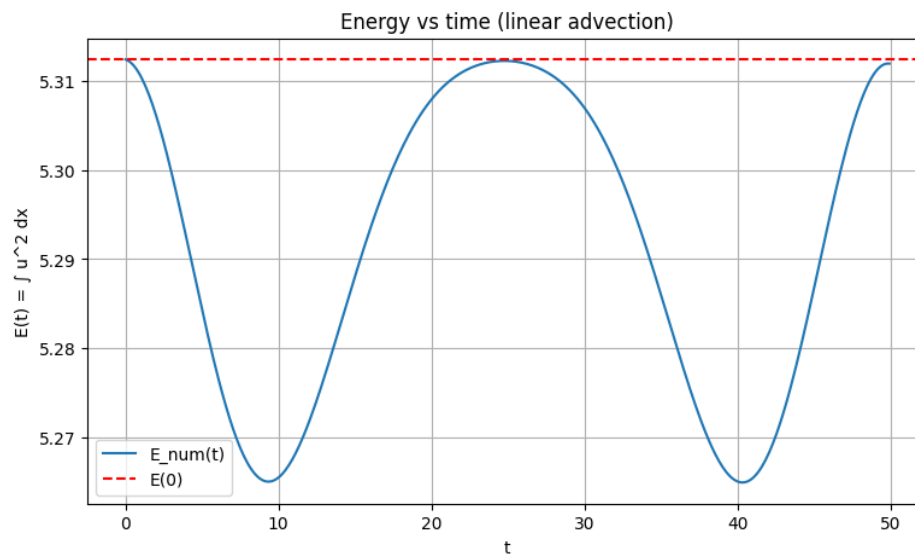
$$\boxed{\frac{dE}{dt} = 0}.$$

The PDE conserves energy, consistent with the Fourier analysis where each mode has constant amplitude.

(ii) Numerical energy evolution. The numerical energy

$$E_{\text{num}}(t) \approx \sum_j u_j(t)^2 \Delta x$$

remains nearly constant, with only tiny oscillations due to truncation error:



This matches the analytical prediction and validates the accuracy of the simulation.

3 Question 3: Diffusion

PDE

We consider the diffusion PDE

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad \nu = 0.1.$$

The domain is subject to periodic boundary conditions:

$$u(0, t) = u(L, t), \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t).$$

(a) Diffusion Stability Limit

CFL restrictions do not normally apply to the diffusion equation; however, we may still examine the discrete form of the PDE to obtain a CFL-like stability condition. Applying a forward difference in time and a centered difference in space (the FTCS scheme) yields the partial difference equation.

We begin with the forward difference in time,

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t},$$

and the centered difference in space,

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}.$$

Equating the two approximations through the diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2},$$

we obtain

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \nu \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}.$$

Introducing the notation

$$u(x_j, t_m) = u_j^m.$$

Therefore, the finite-difference relation becomes

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = \nu \frac{u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)}}{(\Delta x)^2}.$$

Rearranging to isolate $u_j^{(m+1)}$, we obtain

$$u_j^{(m+1)} = u_j^{(m)} + \frac{\nu \Delta t}{(\Delta x)^2} \left(u_{j+1}^{(m)} - 2u_j^{(m)} + u_{j-1}^{(m)} \right).$$

To ensure stability, we examine the term that results from $\frac{\nu \Delta t}{(\Delta x)^2}$. A Von Neumann stability analysis applied to the FTCS scheme yields the condition

$$\frac{\nu \Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$

Solving for Δt gives the diffusion stability limit

$$\Delta t = \frac{(\Delta x)^2}{2\nu} = 5 (\Delta x)^2$$

which we consider the upper bound of our restriction.

(b) Second-Order Finite Difference Scheme

We begin with the Taylor expansions of $u_j(x)$ about x :

$$u_{j+1}(x) = u_j(x + \Delta x), \quad u_{j-1}(x) = u_j(x - \Delta x)$$

which we expand into the series

$$u_{j+1}(x) = u_j(x) + \Delta x u_j'(x) + \frac{\Delta x^2}{2!} u_j''(x) + \frac{\Delta x^3}{3!} u_j^{(3)}(x) + \frac{\Delta x^4}{4!} u_j^{(4)}(x) + \dots$$

$$u_{j-1}(x) = u_j(x) - \Delta x u_j'(x) + \frac{\Delta x^2}{2!} u_j''(x) - \frac{\Delta x^3}{3!} u_j^{(3)}(x) + \frac{\Delta x^4}{4!} u_j^{(4)}(x) + \dots$$

Adding the two expansions gives

$$u_{j+1}(x) + u_{j-1}(x) = 2u_j(x) + \Delta x^2 u_j''(x) + \frac{\Delta x^4}{12} u_j^{(4)}(x) + \dots$$

Subtracting $u_j(x)$, we get

$$u_{j+1}(x) - 2u_j(x) + u_{j-1}(x) = \Delta x^2 u_j''(x) + \frac{\Delta x^4}{12} u_j^{(4)}(x) + \dots$$

Dividing by Δx^2 , we obtain the second-order finite difference approximation

$$\frac{u_{j+1}(x) - 2u_j(x) + u_{j-1}(x)}{(\Delta x)^2} = u_j''(x) + \frac{\Delta x^2}{12} u_j^{(4)}(x) + \dots,$$

where we consider the second-order finite difference scheme to be

$$\boxed{\frac{u_{j+1}(x) - 2u_j(x) + u_{j-1}(x)}{(\Delta x)^2} = u_j''(x)}$$

The term

$$\frac{\Delta x^2}{12} u_j^{(4)}(x)$$

represents the leading-order error term in the finite difference approximation.

We can investigate the validity of the approximation by comparing the approximation values to the expected values. We expect an error that follows an error of Δx^2 as this is our leading-order error term in our approximation.

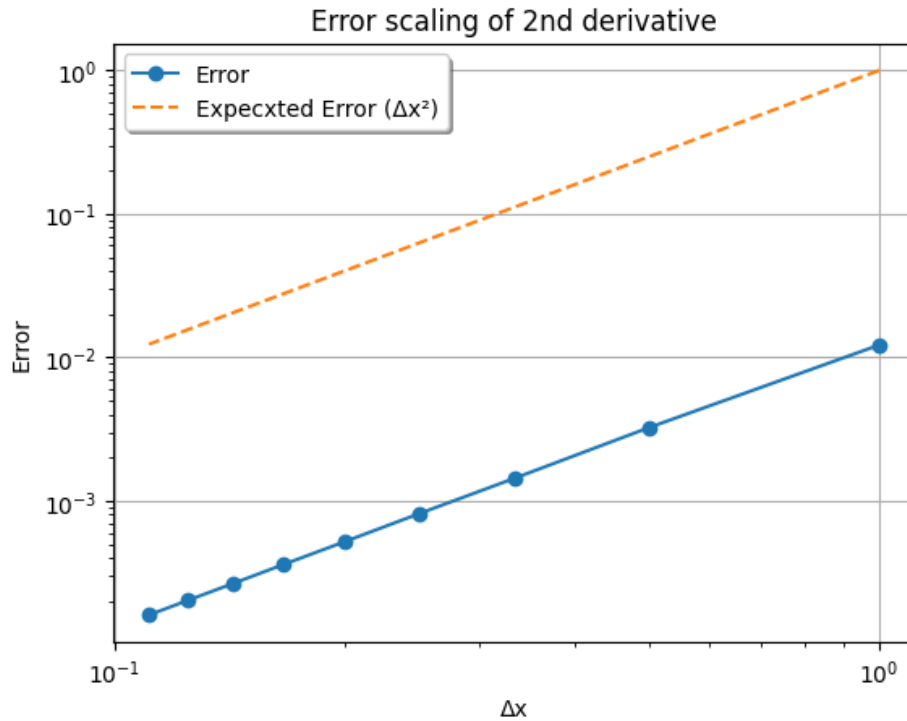
For our analysis we use the function

$$u(x) = \sin\left(\frac{2\pi x}{L}\right)$$

for an expected 2nd derivative being

$$u_{xx} = \frac{-4\pi^2}{L^2} \sin\left(\frac{2\pi x}{L}\right).$$

Comparing the expected derivative values to the approximate we find



Within the graph we can see that the numerical error and the expected error line $\propto \Delta x^2$ are parallel, both having a slope of 2 on the log-log plot. This confirms that the leading truncation error term is Δx^2 , so the centered finite-difference approximation for u_{xx} is second-order accurate. Thus the numerical results match the expected behavior from the Taylor expansion.

(c) Dispersion relation and Fourier interpretation

We use the plane-wave ansatz

$$u(x, t) = e^{i(kx - \omega t)}.$$

Differentiating gives

$$u_t = -i\omega u, \quad u_{xx} = -k^2 u.$$

Substituting into the PDE,

$$-i\omega u = -\nu k^2 u \quad \Rightarrow \quad \omega = -ik^2 \nu.$$

Thus the dispersion relation is

$$\boxed{\omega(k) = -ik^2 \nu},$$

We may rewrite the original solution as

$$u(x, t) = e^{ikx - i(-ik^2\nu)t}.$$

This becomes

$$u(x, t) = e^{ikx - k^2\nu t}.$$

Splitting into imaginary and real parts:

$$u(x, t) = \underbrace{e^{ikx}}_{\text{imaginary}} \underbrace{e^{-k^2\nu t}}_{\text{real}}.$$

This reveals that the ansatz is a product of an imaginary oscillating spatial term and a real temporal decay term. Because ω is purely imaginary, the temporal factor does not oscillate. The solution instead decays exponentially in time according to its wavenumber k . A higher wavenumber would cause a faster decay and a low wavenumber would cause a slower decay.

We can write this in terms of a Fourier series due to the initial conditions being periodic, meaning

$$u(x, t) = \sum_k u_k(0) e^{ikx - k^2\nu t}$$

(d) Numerical method, Parameters, and Simulation Results

We begin with the finite difference approximation for diffusion

$$u_{j+1}^m = u_j^m + \frac{\nu\Delta t}{(\Delta x)^2} (u_{j+1}^m - 2u_j^m + u_{j-1}^m),$$

where we require the stability constraint

$$\frac{\nu\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

Choice of spatial resolution and time step. We are given two modes from the initial condition which gives us two wavelengths:

$$\lambda_1 = \frac{L_x}{2} = 5, \quad \lambda_2 = \frac{L_x}{4} = 2.5.$$

The shortest wavelength is $\lambda_{\min} = 2.5$. We choose the smaller wavelength λ_2 to ensure appropriate spatial resolution. To achieve at least 50 points per wavelength (this allows a

clean resolution for all spatial features), we impose

$$\frac{\lambda_2}{\Delta x} = 50.$$

Solving for Δx :

$$\Delta x = \frac{\lambda_2}{50} = \frac{2.5}{50} = 0.05.$$

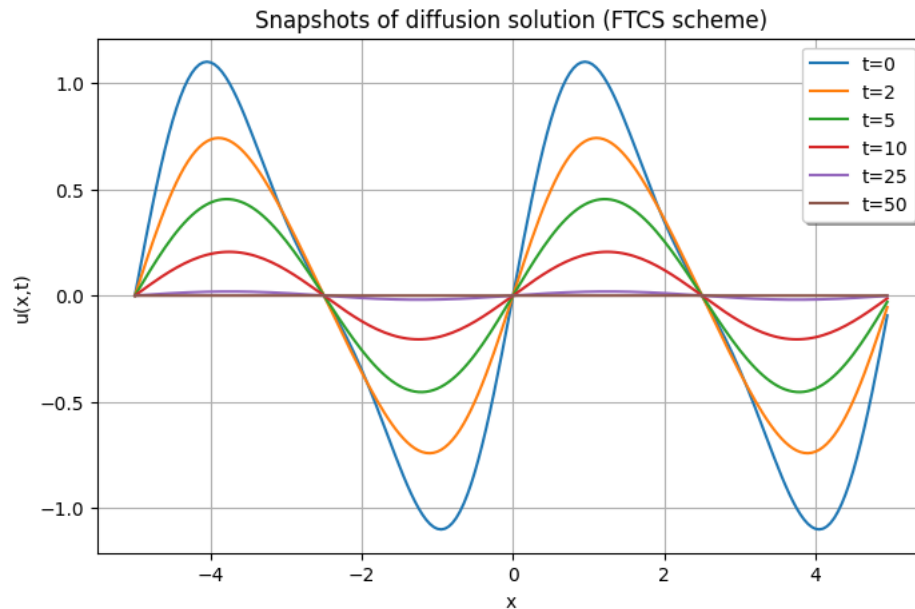
This gives a spatial resolution of

$$N_x = \frac{L_x}{\Delta x} = \frac{10}{0.05} = 200,$$

and a time step of

$$\Delta t = 5(.05)^2 = .0125$$

Qualitative observations. Taking snapshots at several times highlights the key features of the diffusion process. As time evolves, the exponential decay term $e^{-k^2\nu t}$ acts on each Fourier mode, causing the amplitude of $u(x,t)$ to decrease. The higher-wavenumber components decay the fastest, so the solution rapidly becomes smoother and its overall magnitude is steadily damped.



(e) Energy analysis

Define the energy

$$E(t) = \int_{-L_x/2}^{L_x/2} u(x, t)^2 dx.$$

(i) Derivation of ODE for $E(t)$.

Differentiate:

$$\frac{dE}{dt} = \int 2u u_t dx = \int 2u(\nu u_{xx}) dx = 2\nu \int uu_{xx} dx.$$

Integration by parts gives

$$\int uu_{xx} dx = \underbrace{(uu_x) \Big|_{-L_x/2}^{L_x/2}}_{=0 \text{ from BC}} - \int (u_x)^2 dx$$

because the solution is periodic. Thus

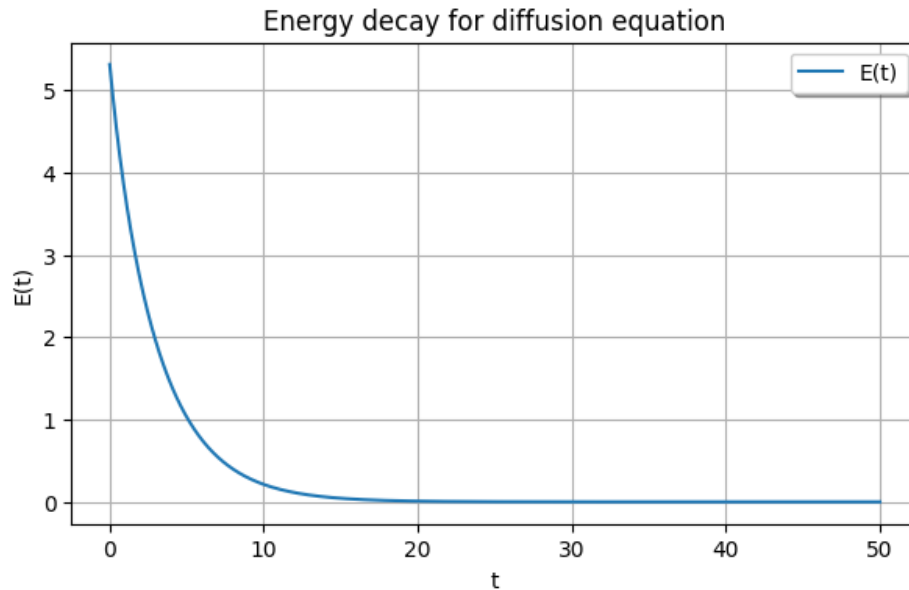
$$\boxed{\frac{dE}{dt} = -2\nu \int \left(\frac{\partial u(x, t)}{\partial x} \right)^2 dx \leq 0},$$

Because the squared term must be greater than 0, the solution must always be negative due to the leading negative sign. This means that the total energy is decreasing over time. This solution is consistent with our finding from our Fourier analysis where each mode is decaying with respect to the term $e^{-k^2 \nu t}$.

(ii) **Numerical energy evolution.** The numerical energy

$$E_{\text{num}}(t) \approx \sum_j u_j(t)^2 \Delta x$$

decays over time from our Fourier analysis.



As seen in the graph, the total energy exponentially decreases in time with no growth or oscillations. This matches our expected energy decay over time from the prior sections.

4 Question 4: Dispersion

We consider the linear dispersive equation

$$u_t = \alpha u_{xxx}, \quad \alpha = 0.1, \quad (1)$$

on the periodic domain $x \in [-L_x/2, L_x/2]$, with $L_x = 10$, over $t \in [0, 50]$. The initial condition is

$$u(x, 0) = \sin\left(\frac{4\pi x}{L_x}\right) + 0.25 \sin\left(\frac{8\pi x}{L_x}\right). \quad (2)$$

(a) Diffusion Stability Limit

We once again can examine the discrete form of our PDE to obtain a CFL-like stability condition by applying a forward difference in time and a centered difference in space (the FTCS scheme).

We begin with the forward difference in time,

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t},$$

and the centered difference in space,

$$\frac{\partial^3 u}{\partial x^3} \approx \frac{u(x + 2\Delta x, t) - 2u(x + \Delta x, t) + 2u(x - \Delta x, t) - u(x - 2\Delta x, t)}{2(\Delta x)^3}.$$

Equating the two approximations through the PDE

$$u_t = \alpha u_{xxx},$$

we obtain

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \alpha \frac{u(x + 2\Delta x, t) - 2u(x + \Delta x, t) + 2u(x - \Delta x, t) - u(x - 2\Delta x, t)}{2(\Delta x)^3}.$$

Introducing the notation

$$u(x_j, t_m) = u_j^{(m)},$$

the finite-difference relation becomes

$$\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} = \alpha \frac{u_{j+2}^{(m)} - 2u_{j+1}^{(m)} + 2u_{j-1}^{(m)} - u_{j-2}^{(m)}}{2(\Delta x)^3}.$$

Rearranging to isolate $u_j^{(m+1)}$, we obtain

$$u_j^{(m+1)} = u_j^{(m)} + \frac{\alpha \Delta t}{2(\Delta x)^3} \left(u_{j+2}^{(m)} - 2u_{j+1}^{(m)} + 2u_{j-1}^{(m)} - u_{j-2}^{(m)} \right).$$

To determine the stability, we examine the term that results from $s = \frac{\alpha \Delta t}{(\Delta x)^3}$. A Von Neumann stability analysis applied to the FTCS scheme yields the condition

$$\frac{\alpha \Delta t}{(\Delta x)^3} \leq 1.$$

Solving for Δt gives

$$\Delta t \leq \frac{(\Delta x)^3}{\alpha} = \frac{(\Delta x)^3}{0.1} = 10 (\Delta x)^3.$$

(b) Second-Order Finite Difference Scheme

To derive the centered, finite difference scheme for second derivatives, we begin with a Taylor series expansion for $u_{i+1}, u_{i-1}, u_{i+2}$, and u_{i-2} out to the fifth-order term. This yields

$$u_{i+1} = u_i + u'_i \Delta x + \frac{1}{2} u''_i \Delta x^2 + \frac{1}{3!} u'''_i \Delta x^3 + \frac{1}{4!} u^{(4)}_i \Delta x^4 + \frac{1}{5!} u^{(5)}_i \Delta x^5 + \dots$$

$$u_{i-1} = u_i - u'_i \Delta x + \frac{1}{2} u''_i \Delta x^2 - \frac{1}{3!} u'''_i \Delta x^3 + \frac{1}{4!} u^{(4)}_i \Delta x^4 - \frac{1}{5!} u^{(5)}_i \Delta x^5 - \dots$$

$$u_{i+2} = u_i + 2 u'_i \Delta x + \frac{(2\Delta x)^2}{2!} u''_i + \frac{(2\Delta x)^3}{3!} u'''_i + \frac{(2\Delta x)^4}{4!} u^{(4)}_i + \frac{(2\Delta x)^5}{5!} u^{(5)}_i + \dots$$

$$u_{i-2} = u_i - 2 u'_i \Delta x + \frac{(2\Delta x)^2}{2!} u''_i - \frac{(2\Delta x)^3}{3!} u'''_i + \frac{(2\Delta x)^4}{4!} u^{(4)}_i - \frac{(2\Delta x)^5}{5!} u^{(5)}_i + \dots$$

Now using a linear combination of the form

$$C_1 u_{i-2} + C_2 u_{i-1} + C_3 u_i + C_4 u_{i+1} + C_5 u_{i+2}.$$

It can be said for the leading term, that the first, second, fourth derivatives, and so on cancel, leaving the third derivative. Since we want our scheme to be a centered derivation, we can take

$$C_1 = -\frac{1}{2}, \quad C_2 = 1, \quad C_3 = 0, \quad C_4 = -1, \quad C_5 = \frac{1}{2}$$

We therefore can rewrite as

$$-\frac{1}{2}u_{i-2} + u_{i-1} - u_{i+1} + \frac{1}{2}u_{i+2} = 2u_i''' \Delta x^3 + \dots$$

Solving for u_i'''

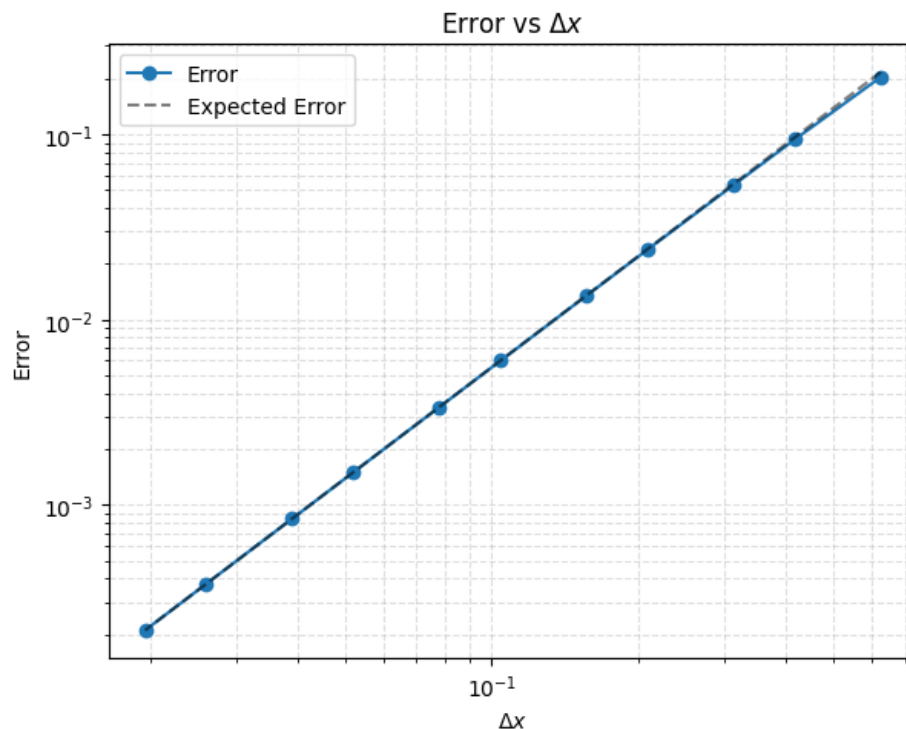
$$u'''(x_i) \approx \frac{-\frac{1}{2}u_{i-2} + u_{i-1} - u_{i+1} + \frac{1}{2}u_{i+2}}{\Delta x^3}.$$

Thus, the final centered, finite difference stencil for the third derivative is

$$\frac{-\frac{1}{2}u_{i-2} + u_{i-1} - u_{i+1} + \frac{1}{2}u_{i+2}}{\Delta x^3} = u'''(x_i).$$

This matches the provided code from the Workbook.

We can now observe the error curve to validate the scheme



The Error vs. Δx plot shows the error decreasing like Δx^2 , which is exactly what we expect from a second-order finite-difference scheme. As Δx^2 gets larger, we find that the error increases. This confirms that our method is behaving correctly and matches the expected accuracy from the previous workbooks.

(c) Dispersion relation and Fourier interpretation

We use the plane-wave ansatz

$$u(x, t) = e^{i(kx - \omega t)}. \quad (3)$$

Differentiating gives

$$u_t = -i\omega u \implies u_x = ik u, \implies u_{xxx} = (ik)^3 u = i^3 k^3 u = -ik^3 u. \quad (4)$$

Substituting into the PDE,

$$-i\omega u = \alpha(-ik^3)u \implies \omega = \alpha k^3. \quad (5)$$

Thus the dispersion relation is

$$\boxed{\omega(k) = \alpha k^3.}$$

which is real for α and k .

Substituting back into $u(x, t)$, we obtain

$$u(x, t) = e^{i(kx - \omega t)} = e^{i(kx - \alpha k^3 t)}. \quad (6)$$

The exponent has no real part, so the amplitude of each Fourier mode is constant in time.

Each mode can be translated with a phase speed

$$c_{\text{phase}} = \frac{\omega}{k} = \alpha k^2. \quad (7)$$

This value is dependent on the wavenumber k . We can therefore conclude that the system is *dispersive*, as expected. Different Fourier components travel at different speeds, this results in less uniform shifts of the waveform over time, compared to Problem 1 for example.

Because a general periodic initial condition can be written as a Fourier series, in relation to k , such that

$$u(x, 0) = \sum_k u_0 e^{ikx} = \sum_k u_0 e^{i(kx - \alpha k^3 t)}. \quad (8)$$

Each Fourier coefficient keeps the same amplitude, while the phase changes according to the dispersive relation of $\omega = \alpha k^3$.

(d) Numerical method, parameters, and simulation results

We solve the dispersive PDE on a periodic grid using a second-order finite difference scheme for the third derivative found in part (b):

$$u_{xxx}(x_j) \approx \frac{-\frac{1}{2}u_{j+2} + u_{j+1} - u_{j-1} + \frac{1}{2}u_{j-2}}{\Delta x^3}. \quad (9)$$

Similar format to previous problems, this yields

$$u_j^{n+1} = u_j^n + \alpha \Delta t \frac{-\frac{1}{2}u_{j+2}^n + u_{j+1}^n - u_{j-1}^n + \frac{1}{2}u_{j-2}^n}{\Delta x^3}. \quad (10)$$

With CFL number

$$C = \frac{\alpha \Delta t}{\Delta x^3} \quad (11)$$

Choice of spatial resolution. The initial condition contains two wavelengths:

$$\lambda_1 = \frac{L_x}{2} = 5, \quad \lambda_2 = \frac{L_x}{4} = 2.5.$$

The shortest wavelength is $\lambda_{\min} = 2.5$. With $N_x = 128$,

$$\Delta x = \frac{L_x}{N_x} = 0.07912, \quad \frac{\lambda_{\min}}{\Delta x} \approx 32 \text{ points/wavelength},$$

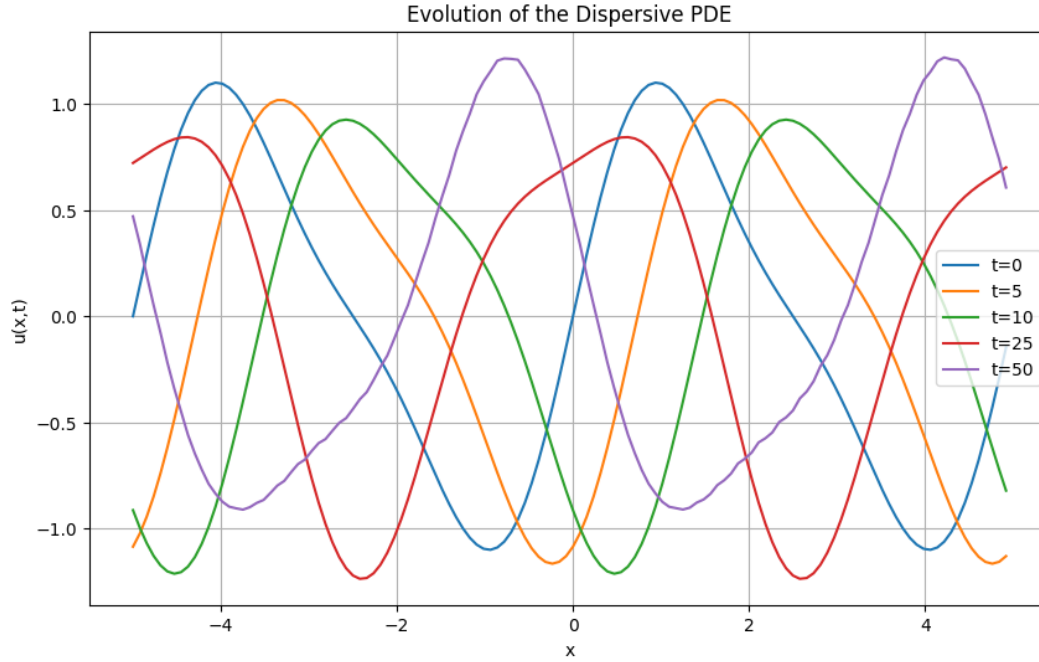
which cleanly resolves all spatial features. One could use $N_x = 256$ to smoothen the graph further, but it was found that overall it did not significantly improve the curve while adding a large amount of time to the generation of the code.

Choice of time step. We pick $C = 0.0005$ for a CFL. It was found that with a CFL > 0.0005 , that the code could not be properly generated. An overflow error results in only half the time snapshots being plotted. Further, values < 0.0005 were found to take much longer to generate with minimal improvement to the graph ($C = 0.0004$ had a generation time of 25.4 minutes). Choosing $C = 0.0005$ ensured the scheme remained fully stable for the entire simulation window and prevented the overflow seen in prior attempts. This small CFL value allowed us to generate all outputs cleanly and obtain a physically meaningful evolution of our solution. Thus

$$\Delta t = C \frac{\Delta x}{\alpha} = 0.0005 \frac{0.0390625}{0.1} = 1.953125 \times 10^{-4} \approx 1.95 \times 10^{-4}.$$

Qualitative observations. Snapshots of the numerical solution at several times show a strong dispersive spread. As we expect, the initial condition contains two waveforms that

travel with different phase speeds dependent on wavenumber k . Therefore, their relative phase changes in time, causing the waveform to distort despite a constant amplitude being held over time. This is in agreement with part (a), as we expect the waveform to be dispersive but still preserve energy over time.



(e) Energy analysis

Define the energy

$$E(t) = \int_{-L_x/2}^{L_x/2} u(x, t)^2 dx. \quad (12)$$

(i) Derivation of ODE for $E(t)$.

Differentiate with respect to time:

$$\frac{dE}{dt} = \frac{d}{dt} \int_{-L_x/2}^{L_x/2} u(x, t)^2 dx = \int_{-L_x/2}^{L_x/2} 2u u_t dx. \quad (13)$$

Apply the dispersion PDE $u_t = \alpha u_{xxx}$

$$\frac{dE}{dt} = 2\alpha \int_{-L_x/2}^{L_x/2} u u_{xxx} dx. \quad (14)$$

Using integration by parts with $f = u \implies df = u_x dx$, $dg = u_{xxx} dx \implies g = u_{xx}$, the solution yields

$$\frac{dE}{dt} = 2\alpha \left([u u_{xx}]_{-L_x/2}^{L_x/2} - \int_{-L_x/2}^{L_x/2} u_x u_{xx} dx \right). \quad (15)$$

Since the solution is periodic, the boundary term vanishes $[u u_{xx}]_{-L_x/2}^{L_x/2} = 0$. For the remaining integral we use

$$\int u_x u_{xx} dx = \frac{1}{2} [u_x^2]_{-L_x/2}^{L_x/2} = 0, \quad (16)$$

Hence we find that

$$\boxed{\frac{dE}{dt} = 0}$$

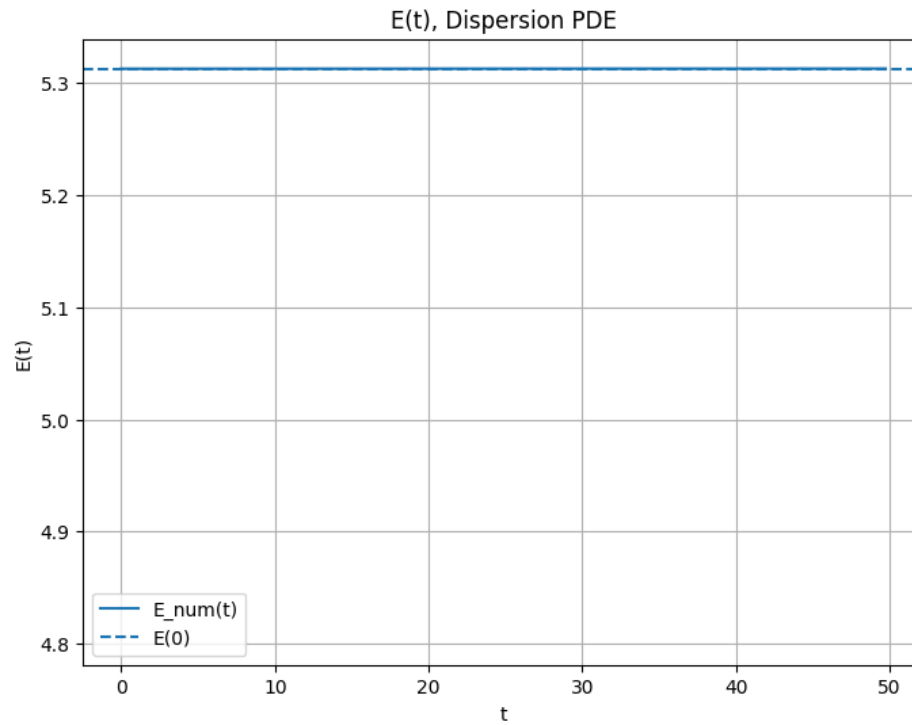
So the PDE conserves the energy, which is consistent with the Fourier analysis where each mode has constant amplitude in time.

(ii) Numerical energy evolution.

The numerical energy

$$E_{\text{num}}(t) \approx \sum_j u_j(t)^2 \Delta x$$

remains nearly constant, with only tiny oscillations due to truncation error. The equation above is purely a way to sum $E(t)$ for the purposes of coding a plot. Similar to Problem 1, we find that there is no gain or loss of energy. Below is the plot for $E(t)$



Because the dispersion equation nearly completely conserves energy, the curve lies directly on top of the provided $E(0)$ line. The plot matches our analytical prediction and validates the accuracy of the simulation.

Question 5: Nondimensionalization – Comparing Terms

We consider the viscous Burgers' equation,

$$\partial_t u(x, t) = -u \partial_x u + \nu \partial_{xx} u.$$

The Burgers' PDE was initially seen in Project 2, where we determined that there are nonlinear advection and diffusion terms. In this problem, we aim to compare the relative importance of these over time through nondimensionalization.

(a) Nondimensionalization of the PDE

To non-dimensionalize the PDE, we begin by considering the canonical length-scale L , velocity scale U , and an advective time-scale, $T = \frac{L}{U}$, provided. We can then introduce three non-dimensional variables, such that

$$x = L x', \quad u = U u', \quad t = T t' = \frac{L}{U} t'$$

We can then rewrite the derivatives as

$$\partial_x = \frac{1}{L} \partial_{x'} \implies \partial_{xx} = \frac{1}{L^2} \partial_{x'x'} \quad \partial_t = \frac{U}{L} \partial_{t'}.$$

Substituting into Burgers' equation yields

$$\frac{U}{L} \partial_{t'} U u' = -(U u') \left(\frac{1}{L} \partial_{x'} U u' \right) + \nu \left(\frac{1}{L^2} \partial_{x'x'} U u' \right).$$

Or more simply

$$\frac{U^2}{L} \partial_{t'} u' = -\frac{U^2}{L} u' \partial_{x'} u' + \frac{\nu U}{L^2} \partial_{x'x'} u'$$

(b) Comparing relative magnitudes: Reynolds number

The time derivative and nonlinear terms both have the coefficient

$$\frac{U^2}{L},$$

The diffusion term carries the coefficient

$$\frac{\nu U}{L^2}.$$

The ratio of the nonlinear term to the diffusive term is

$$\frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = Re.$$

This quantity is known as the Reynolds number.

If we divide through by Reynolds number, the PDE becomes

$$\partial_t u' = -u' \partial_{x'} u' + \frac{1}{Re} \partial_{x'x'} u'.$$

From the analysis above, we can conclude that if Re is very large, the nonlinear steepening dominates diffusion. This results in steeper gradients and a more turbulent curve. With the diffusion term overpowered, the smooth behavior is severely decreased. If Re is very large, the diffusion will dominate. This will result in much smoother curve that dampens the gradient significantly quicker.

(c) Length scale at which diffusion dominates

Using the Reynolds number, we can solve for the characteristic length scale L where the diffusion component becomes comparable to nonlinear component ($Re = 1$). We thereby write

$$L = \frac{\nu}{U}.$$

From inspection, we can conclude that the velocity U increases when the kinetic energy increases. From the equation above, we see that as U increases, L decreases. A smaller value of L results in the need for a higher spatial resolution (smaller Δx), and a higher amount of grid points (increase in N_x).

This explains why a higher resolution is required for Burgers' numerical simulations, specifically at higher energy levels.

(d) Spectral behavior and characteristic length-scale

For our Reynolds number we can solve for the characteristic length-scale L' corresponding to the given values of Re :

$$L' = \frac{Re \nu}{U}.$$

The Burgers' data is set to a fixed viscosity of $\nu = 0.00078125$. We can then take U to be the root-mean-square velocity. We can then compute the numerical solution at each time at each provided Reynolds numbers

$$Re \in \{1, 2, 5\},$$

this gives a time-dependent characteristic length-scale

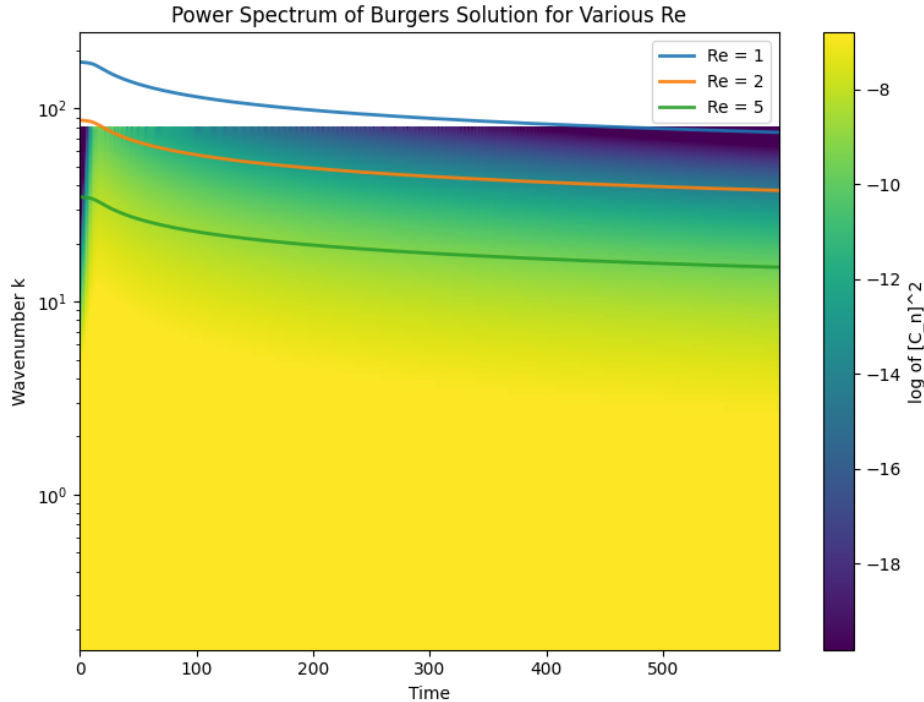
$$L'(t) = \frac{Re \nu}{U_{rms}(t)}.$$

where $U_{rms}(t)$ is the root-mean-square velocity.

We can convert the length-scale into corresponding wavenumbers of $2\pi/\ell$ to plot on a wavenumber axis

$$k(t) = \frac{2\pi}{L'(t)}.$$

We can then create a colour plot of spectral power, where the two axis are wavenumber and time. We can also plot three curves of $k(t)$ for the given $Re = 1, 2, 5$. These lines indicate, at each time, the wavenumber at which diffusion becomes comparable to the nonlinear steepening for each Reynolds number.



Note: The white space at the top of the graph is due to a limit in the number of wavenumbers that can be generated from the Burgers' data.

From the plot, we can conclude that when the length-scale ℓ is smaller than $L'(t)$ indicates that the value lies in the diffusion dominated region (where diffusion dominates over the nonlinear advection at that scale). As a result, the spectral power at those wavenumbers is rapidly damped. Energy is therefore lost quickly in this region. Conversely, when ℓ is larger than $L'(t)$, the nonlinear region has the dominating effect.

Over time, it can be seen through the plot that $L'(t)$ increases. Recall that $L'(t)$ is in the denominator for $k(t)$, therefore the curves move toward smaller wavenumbers over time. This matches the evolution of spectral power in the color plot. At early times, most of the energy sits at high wavenumbers. This indicates the steep gradients (particularly for higher k) mentioned earlier. Over time, these wavenumbers decay more rapidly. This results in a high amount of energy being concentrated at lower wavenumbers, as a much larger and smoother plot forms. Therefore, as $U_{rms}(t)$ decreases over time, $L'(t)$ increases and thereby $k(t)$ decreases, which matches the observations of plot.